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APOLLO EXPERIENCE REPORT - ONBOARD NAVIGATIONAL AND ALIGNMENT SOFTWARE

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16. Abstract

The onboard navigational and alignment routines used during the nonthrusting phases of an Apollo mission are discussed as to their limitations, and alternate approaches that have more desirable capabilities are presented. This discussion includes a more efficient procedure for solving Kepler's equation, which is used in the calculation of Kepler's problem and Lambert's problem; and a sixth-order predictor (no corrector used) scheme with a Runge-Kutta starter is recommended for numerical integration. Also evaluated are the extension of the rendezvous navigation state to include angle biases and the use of a fixed coordinate system. In general, the discussion reflects the advancement of the state of the art since the design of the Apollo software.

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APOLLO EXPERIENCE REPORT

ONBOARD NAVIGATIONAL AND ALIGNMENT SOFTWARE

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SUMMARY

The current limitations of the onboard navigational and alignment software are discussed, along with more desirable capabilities and alternate approaches that are available. The onboard navigational and alignment software includes programs for free-flight prediction, rendezvous navigation, orbital navigation, cislunar navigation, and alignment of the inertial measurement unit and support routines (e.g., lunar and solar ephemerides and planetary inertial orientation).

INTRODUCTION

The onboard navigational and alignment routines discussed in this report are used during the nonthrusting phases of an Apollo mission. The basic objective of the navigational system is to maintain estimates of the position and velocity of the command and service module (CSM) and lunar module (LM). Navigation is accomplished by extrapolating the state vectors by means of the coasting integration routine and then updating this estimate by processing tracking data by means of a recursive navigational method.

The CSM guidance and navigation system uses (1) range data from the very-high-frequency (vhf) ranging device and (2) angular data from the scanning telescope and from the sextant. The LM primary guidance and navigation system uses rendezvous radar (RR) tracking data (angles, range, and range rate). In addition, the LM system has an alignment telescope for platform alignments.

These navigational data are incorporated into the state vector estimates by means of the measurement-incorporation routine. This routine computes deviations to the state vector based on the tracking geometry and the statistics of the state vector history.

In this report, the limitations of these routines are discussed, and alternate approaches with more desirable capabilities are presented. In general, the comments reflect the advancement of the state of the art since the design of the Apollo software.

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An efficient Cartesian coordinate transformation method is described in the appendix written by Paul F. Flanagan of the NASA Manned Spacecraft Center and Samuel Pines of Analytical Mechanics Associates.

DISCUSSION

Navigational System

The basic objective of the navigational system (maintaining estimates of position and velocity vectors for the CSM and LM) is accomplished by extrapolating the state vector by means of the coasting integration routine. The procedure used by this routine is to extrapolate the state vector by means of Encke's method of differential accelerations, in which only deviations from conic motion are integrated numerically. This approach is sound and represents the current state of the art. However, subtle improvements are now available for determining the conic motion and numerically integrating the deviations.

Even with accurate state vector extrapolation, initial-condition errors grow to an intolerable size. Thus, it is necessary to periodically obtain additional data to modify the state vectors. These modifications are computed from navigational data obtained from sensor measurements. The nature of the measurement sensors requires significant crew interface. The introduction of human errors into the navigational system can be minimized in future projects by using a more automated sensor system; this step not only would improve basic navigational data but also could partially eliminate or reduce the need for premission scheduling of sightings.

When a measurement is made, the best estimate of the state vector is extrapolated to the measurement time. From this estimate, it is possible to compute an estimate of the quantity measured. When this computed measurement is compared with the actual measured quantity, the difference is used to update the state vector. This difference or deviation corrects the state as a linear multiplier of a weighting vector (which in turn is a function of the geometry, the assumed measurement uncertainties, and the state uncertainties). The major problem for this area of navigation has been error-transition-matrix saturation. This matrix allows reasonable corrections in the beginning but rapidly reduces the allowable correction until a point is reached when no useful corrections are permitted. This problem has been circumvented by constant reinitialization of this matrix, requiring extensive premission navigational analysis to determine when this event should occur. Nonnominal navigation could have reduced the effectiveness of this scheduling.

The Apollo guidance computer was designed with a 15-bit word length. This computer is a fractional machine (all numbers in the computer are less than one); thus, to store or use a number with a true value greater than one, suitable scaling is necessary. As a result, parameters with a large dynamic range are scaled so as to optimize the capabilities of the computer. Invariably, however, the extremes of these numbers are compromised. This limitation has produced errors of as much as 240 feet down range during coasting integration. The down-range errors have the same effect as increasing the errors on the rendezvous tracking data. In the following paragraphs, the various routines of the navigational system, their limitations, and recommendations for improvements are discussed.

Coasting Integration Routine

The coasting integration routine is the subroutine that — when given initial time, position, and velocity coordinates — computes the position and velocity of the vehicle at a specified time. This time may be either before or after the time of the particular initial state vector. This is the basic routine used in the navigational and guidance programs; therefore, the speed and accuracy of the onboard program are limited by the performance of this routine.

Coasting integration is accomplished by using Encke's method of differential accelerations. With this technique, the motion of the vehicle is assumed to be dominated by the conic orbit that would result if the spacecraft were in a central force field. Then, only the deviations from conic motion are integrated numerically. The nature of this method necessitates a discussion of coasting integration in two parts, namely, conic solution and numerical integration of disturbing accelerations.

Conic method. - The conic method is used in the solution of five conic problems. Three of the conic problems (Kepler, time theta, and time radius) are initial-value problems, and two (Lambert and reentry) are boundary-value problems (ref. 1). The numerical integration formulation requires considerable computer time. For example, the return-to-earth maneuver-guidance program (P-37) requires approximately 15 to 30 minutes to compute a solution.

Because much of the computer time is spent solving Kepler's equation, which is used in the calculation of Kepler's problem and Lambert's problem, a more efficient procedure (such as that proposed in ref. 2) would have been useful. To solve this equation, it is necessary to sum a special trigonometric series. The proposed formulation would modify the method used for generating and summing the series. The series currently used requires special procedures for integration over 2π (and additional terms for special cases) and is slower to converge than the proposed method. In addition, the procedure described in ref. 2 for solving Lambert's problem is more efficient. This method avoids slowly converging inverse trigonometric series and would decrease the required computer time. Other advantages of the proposals for solving Kepler's problem and Lambert's problem are an improvement in storage requirements, the accuracy of the computations, the reliability of the convergence, and computer time.

Numerical integration method. - The complexity of the numerical evaluation of the equations of motion is the largest single item affecting the machine time required for each integration step. The integration method used is a third-order Runge-Kutta scheme developed by Nystrom. This method requires three entries into the lengthy derivative routine for each integration step. For a rapid computing cycle, it is imperative to minimize the number of entries into the derivative routine.

Backward difference schemes have the advantage of requiring only one entry per integration step for prediction methods and a maximum of two entries per integration step for predictor-corrector methods. Another advantage of backward difference schemes is that they enable the use of a much larger integration step than the third-order Runge-Kutta scheme now in use. The limiting considerations of the step size are (1) the truncation error in the Taylor series used to evaluate the derivative and

(2) the machine digit word length, which controls the round-off error. The best step size is provided when the magnitude of the truncation error is the same as that of the round-off error.

Because the magnitude of the truncation error decreases with the number of terms, the limitation on the step size is the correlation of the number of terms and the round-off error. The major source of round-off error is the bias in the round-off digit of the integration coefficients. These coefficients can be kept as exact integers if the number of terms in the backward table is chosen as shown in the following table.

Computing word length, digits	Backward difference predictor (no corrector used), number of terms
5	6
6	7
7	7
8	8
9	9
10	10
11	11

For the numerical integration by the onboard computer with eight-digit arithmetic, a sixth-order predictor (no corrector used) scheme (ref. 3) is recommended. The reference 3 predictor is self-started, but for multiple starts during a mission, a Runge-Kutta starter is recommended. Although an eighth-order scheme could be accommodated while still maintaining the integration coefficients as exact integers (see preceding table), the computer space limitations led to the sixth-order recommendation. Thus, a Runge-Kutta starter would be required only for building the table. Although the backward difference scheme will prove efficient over a relatively long computing arc, the scheme cannot be recommended for use in a Kalman sequential point-by-point update mode with observations.

For the navigation update mode, it is recommended that a less complex method be considered to predict the state to the next observation with an extra error source added to the W-matrix to account for unmodeled errors in the dynamic model.

The error transition matrix. - The position and velocity vectors maintained by the computer with numerical integration are only estimates of the true values. As part of the navigational technique, it is also necessary to maintain statistical data in the computer to aid in processing navigational measurements. To accomplish this, a correlation matrix is defined that represents probable uncertainties in the state and the correlation of these uncertainties. For convenience, a matrix called the error transition matrix (or square root of the correlation matrix) is defined, and this matrix must be maintained by the computer.

Extrapolation of the error transition matrix in the Apollo guidance computer is made by direct numerical integration of as many as nine vector differential equations. This method is costly because most of the time spent in this matrix propagation is in the derivative routine (that portion of the logic that numerically evaluates the integration equations (equations 2.2.35 of ref. 1)).

Reference 2 contains an alternate method for state error propagation. This analytic method, sometimes called the mean conic method, estimates errors at future times by introducing predicted errors at the initial time and propagating these errors along the average conic. This average or mean conic is defined by the initial and final state estimates and by the transfer time. The method involves no numerical integration or iteration and requires the definition of only four indexed analytic partials.

Rendezvous Navigation (P-20)

Rendezvous navigation has been excellent in the Apollo Program; however, if large down-range errors occur in the location of the non-updated vehicle, the CSM rendezvous navigation is degraded. The CSM rendezvous navigation degradation occurs because the capability to solve for angle biases in the CSM filter was not provided, due to the small sextant biases; however, the sextant line-of-sight measurements are made with respect to inertial space and a down-range error. In the non-updated vehicle, position appears as an angle bias. Because rendezvous navigation is a relative problem, it is sufficient that the LM and CSM state vectors be accurate relative to each other, even though the inertial accuracy of each state is poor. The fact that a relative problem is being solved enabled the software designers to design the program to update or solve for the state vector of only one of the two vehicles. Another way to account for the downtrack error is to solve for the state vector of both vehicles; however, this procedure requires too much computer time and is not necessary because the relative problem can be solved in all cases if the solution for the angle biases is part of the solution vector.

Anomalous simulation results.- During the navigational planning and analysis for Apollo 10, two interesting results were obtained from simulations of the onboard rendezvous navigation of the CSM and LM.

1. Although the LM RR is less accurate in its measurements of line of sight and range than is the CSM sextant and vhf ranging, the LM guidance computer (LGC) often obtained a superior determination of the relative state (producing a more accurate rendezvous).

2. The method of immediately obtaining a set of "good" estimates of the RR angle biases (by rolling the LM 180° between the first two data sets) resulted in inferior accuracy compared to the method of allowing these estimates to be determined throughout the rendezvous sequence.

Explanation of results.- As a result of reconsideration of the problem, the anomalous results were attributed to a down-range error in the state vector of the non-updated vehicle.

For the onboard formulations in both vehicles, it is assumed that the state of one vehicle is perfectly known; therefore, sensor residuals are used to correct the other vehicle state vector. Because line-of-sight measurements are made with respect to inertial space in a near-circular orbit, down-range errors in the position of the non-updated vehicle act as angle biases in the filter. (If all navigational measurements and rendezvous maneuvers were made with respect to the local (orbital plane) coordinates, such errors would not be significant; however, the mechanization of such measurements is not feasible.) This apparent bias "drives" the filter away from the correct solution, resulting in state estimates with poor error-propagation characteristics.

During the design of the filters for onboard navigation, the RR angle biases were expected to be rather large; therefore, they were included in the LGC solution vector. Apparent angle biases caused by down-range errors are attributed by the LGC to physical angle biases and do not severely degrade the state vector of the updated vehicle. The command module computer (CMC) does not provide such an attribution for these biases; therefore, they degrade the state vector solution. Rolling the LM to quickly determine the biases degrades the solution because the filter "closes out" subsequent significant updates to the angle biases, and the down-range error is not recovered as such a bias. (Reinitializing the filter in order to reopen it to further bias updates, although retaining the bias estimates, would theoretically provide excellent performance; however, the RR angle biases have tended to be small enough in practice that such a procedure is not worthwhile.)

Command module computer change. - Studies were undertaken to evaluate the usefulness of expanding the CMC solution vector to solve for the down-range error, either explicitly or as angle biases; favorable results were obtained. Simulations indicated that the approach was valuable and also led to a slight modification of the original equations.

Examination of the equations and the existing CMC program structure led to new estimates of additional storage requirements, namely, 50 words of fixed memory and two or three words of erasable memory. A Program Change Request (PCR) was written for consideration by the Apollo Software Control Board (SCB).

During the evaluation of the proposal, the contractor for the design of the primary guidance, navigation, and control system pointed out that, in the Apollo CMC programs, a reallocation of erasable memory assignments would be required because the rendezvous targeting routines shared erasable memory with cells used for W-matrix storage in the 9-D mode. The SCB did not approve the PCR primarily because the CMC rendezvous navigation program is used as a backup capability on lunar flights. Of course, in earth orbit, down-range errors are less important; but the error that does occur, when taken in combination with the inertial measurement unit (IMU) and sextant biases, produces poor accuracy for some rendezvous profiles.

Orbital Navigation (P-22)

Although a limited capability to perform orbital navigation exists in the software, it has not been a requirement in the Apollo Program. However, P-22 was used to gather time-mark data for Manned Space Flight Network (MSFN) orbital navigation evaluation, lunar mapping, and descent targeting. (A new program (P-24), which removed the liabilities of P-22, was added to the software for Apollo 14 and subsequent flights. The liabilities of P-22 are the lack of acquisition assistance and the five-mark limitation.)

Cislunar Navigation (P-23)

Cislunar navigation was satisfactory for the Apollo missions. A problem occurred on the Apollo 14 mission when the incorrect lunar horizon was used as a target. Procedures have been developed to eliminate this error, which occurred as a result of the moon's being nearly fully sunlit. Software modifications such as solutions for the altitude of the horizon locator and the sextant bias could be evaluated.

Inertial Measurement Unit Alignment

The IMU alignment programs could possibly be improved in the area of transformation computation. The current 3×3 matrix product-transformation procedure could be replaced by the vector operation discussed in the appendix to this report. However, the only significant problem is in the area of the hardware-software incapability experienced in the LM. The LM IMU alignment system is sufficiently accurate for a safe landing but has only marginal accuracy for precision landing.

The LM alignment problem resulted from the fact that the sighting instrument (the alignment optical telescope (AOT)) is much less accurate than the IMU. The system could have been improved by a better instrument (such as the star tracker or the sextant) or by increasing the number of stars tracked and by averaging data. That is, instead of sighting on only two stars and defining the reference stable member matrix (REFSMMAT), three to five stars could have been sighted in an identical manner (three marks per star). The stars would have been properly chosen as a function of their relative position with respect to one another. Some simple type of averaging (such as a least squares approximation for defining REFSMMAT) that would minimize the error in this transformation could then be accomplished. This averaging technique could be used in the onboard computer to determine the LM position on the moon.

Support Routines

Coordinate systems.- An additional difficulty encountered in the manned space flight program is the use of unconventional coordinate systems. The system based on the X-axis through Greenwich at midnight on the day of launch was used in Project Mercury and in the Gemini Program. The Nearest Besselian Year (NBY) system was used in the Apollo Program. The primary advantage of the NBY coordinate system is that the transformation from earth-fixed coordinates to inertial coordinates is a rotation instead of the 3×3 transformation required for the standard fixed-coordinate systems. However, in retrospect, this small saving in computer time is offset by the complex effort required to convert the ephemeris data to the NBY system and by the necessity to remake the computer "ropes" for each Besselian year. This revision is necessary because the fixed memory contains data that are NBY dependent. In general, the use of a fixed-coordinate system would simplify the overall software problem. The fixed-coordinate system should probably be the 1950.0 system, in which the basic lunar-solar-planet ephemerides are located.

The NBY system was established for the following reasons:

1. The precession of the pole of the earth can be neglected when computing the earth-oblateness accelerations.

2. The transformation to earth-fixed coordinates may use small-angle approximations.

3. The NBY coordinate system is widely used and is one for which information is available from the Nautical Almanac Office.

Although these reasons seem valid, they are not decisive. The effects of precession may be accounted for easily, the small-angle approximations are valid over extended periods, and the Nautical Almanac Office does not publish the data until after they are needed (necessitating an annual request to that office for prepublication data).

Modifications needed for use of a constant system. - The following are the Apollo guidance computer modifications that would be required if a fixed-coordinate system were adopted.

Lunar-solar ephemerides: To use the lunar-solar ephemerides, no problem exists for the CMC because the data are stored entirely in an erasable memory; the generation program may be used with little or no change. For the LGC, a minor change would improve the accuracy and extend the usefulness to several years for the lunar ephemerides. The root mean square of the error in location of the moon was approximately equal to the diameter of the moon (0.5°), and the maximum error during a 1-year span was greater than 1° (17 milliradians). A plot of the lunar ephemerides error was made for in-plane and perpendicular-to-plane locations. The in-plane error was an irregular function but seemed to have a period of 14.5 to 15.0 days. The error perpendicular to the plane was a regular function with a period of 32 days.

The error in the lunar orbit plane could be corrected by an additional term in the expression for the longitude of the moon. The error perpendicular to the orbit plane can be corrected by modifying the earth-mean unit vector to include a final Y-axis rotation. These additional terms correspond approximately to the truncated terms of Brown's series for the position of the moon.

Earth orientation: The small-angle approximations for earth orientation are valid for several years; to use the true earth pole in the oblateness perturbation computation, a small change to $\cos \phi$ is required (requiring one word of fixed memory in both CMC and LGC, but four words of erasable memory in the LGC). The change in the oblateness computation could probably be omitted from the LGC, however.

The diurnal rotation of the earth builds up an error in the Apollo guidance computer of approximately 30 ft/yr because of round-off errors. If this amount is considered excessive, moving constants to erasable memory would permit sufficient correction.

Lunar orientation: No problem exists with use of lunar orientation data, because all residuals may be collected into the lunar "libration vector," which is part of the erasable load.

Star table: Most of the stars in the star table have small proper motions; the vectors are accurate to 5 seconds of arc over periods of ± 3 years. Occasional updating would be required.

Transformations. - A more efficient Cartesian coordinate transformation method that precludes matrix manipulations is discussed in the appendix.

CONCLUDING REMARKS

Apollo mission experience has shown that the basic onboard navigational and alignment software is adequate for lunar landing missions. However, since its inception, the advancement of the state of the art has presented alternate approaches with more desirable capabilities. These new techniques involve more automated sensor systems; improved mathematical functions for numerical integration, conic calculations, and state error propagation; a larger solution state during rendezvous navigation; averaging techniques for star sightings; a more standard basic coordinate system; and an improved transformation algorithm.

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REFERENCES

1. Anon.: Guidance System Operations Plan for Manned CM Earth Orbital and Lunar Missions Using Program COLOSSUS 2D (COMANCHE 72). Section 5, Guidance Equations (Rev. 9). Report R-577, Massachusetts Institute of Technology Instrumentation Laboratory, Nov. 1969.

2. Pines, S.: Formulation of the Two-Body Problem for the On-Board Computer. Report 68-14, Analytical Mechanics Associates, June 1968.

3. Pines, S.; Lefton, L.; Levine, N.; and Whitlock, F.: A Variable Order Interpolation Scheme for Integrating Second Order Differential Equations. Report 69-16, Analytical Mechanics Associates, July 1969.

APPENDIX

AN EFFICIENT METHOD FOR CARTESIAN COORDINATE TRANSFORMATIONS

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INTRODUCTION

An efficient alternate method used to perform Cartesian coordinate transformations is derived. This method replaces the 3×3 matrix product-transformation procedure by a simpler vector operation that requires less computer storage and less computer execution time. Also, for applications in which interpolation routines are applied to stored transformation matrices, this method is more efficient because it requires the interpolation and storage of only four elements, which results in a more accurate final computation.

CARTESIAN COORDINATE TRANSFORMATIONS BY MEANS OF RIGID VECTOR ROTATIONS

A 3×3 transformation matrix A is given that causes the Cartesian coordinate system I to be transformed to the Cartesian coordinate system II. Thus, any vector R_I in the original coordinate system is carried into the second coordinate system by

$$R_{II} = AR_I \tag{1}$$

The same transformation can be achieved by a rigid rotation of R_I about a unit vector N through some angle w. The resultant vector equation is

$$R_{II} = R_I \cos w + NR_I(1 - \cos w)N + \sin w \; NR_I \tag{2}$$

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If the transformation matrix is required, it can be generated from the unit vector N and the angle w by the following equation.

$$A = I \cos w + (1 - \cos w)NN^T + \sin w \, Nx \tag{3a}$$

where

$$Nx = \begin{bmatrix} 0 & -N_3 & N_2 \\ N_3 & 0 & -N_1 \\ -N_2 & N_1 & 0 \end{bmatrix} \tag{3b}$$

The vector N and the trigonometric function cos w are computed from the transformation matrix as follows.

$$\cos w = \frac{\text{trace } A - 1}{2} \tag{4a}$$

$$\sin w = \sqrt{1 - \cos^2 w} \tag{4b}$$

$$N = \left[\left(a_{32} - a_{23}\right)^2 + \left(a_{13} - a_{31}\right)^2 + \left(a_{21} - a_{12}\right)^2 \right]^{-1/2} \begin{bmatrix} a_{32} - a_{23} \\ a_{13} - a_{31} \\ a_{21} - a_{12} \end{bmatrix} \tag{4c}$$

DERIVATION OF THE EQUIVALENT TRANSFORMATION EQUATION

It can be shown that the rigid-rotation matrix may be written as the sum of a symmetric matrix and a skew-symmetric matrix. The symmetric matrix terms of equation (3a) are

$$A \text{ sym} = I \cos w + (1 - \cos w)NN^T \tag{5a}$$

The skew-symmetric terms are given by

$$A \text{ skew sym} = \sin w \; Nx \tag{5b}$$

The trace of A is given by the trace of its symmetric part; thus

$$\text{trace } A = 3 \cos w + (1 - \cos w)\left(N_1{}^2 + N_2{}^2 + N_3{}^2\right) \tag{6}$$

Because N is a unit vector, equation (4a) is obtained.

A rigid rotation about N through an angle w greater than π may be replaced by a rigid negative rotation about the same vector N through $2\pi - w$; therefore, the angle w can be restricted to the first or second quadrants, and equation (4b) results.

To obtain the components of the vector N, note that the skew-symmetric part of A is given by

$$A \text{ skew sym} = \frac{A - A^T}{2} \tag{7}$$

This expression must be equal to the skew-symmetric part of A(N, w) so that

$$\frac{A - A^T}{2} = \sin w \; Nx \tag{8}$$

It follows that because N is a unit vector, equation (4c) is obtained.

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